

PERTH MODERN SCHOOL

YR11 MATHEMATICS SPECIALIST – 2019

TEST 1 – Reasoning & Permutations



PERTH MODERN SCHOOL  
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NAME: Solutions.

DATE: \_\_\_\_\_

Teacher's Name \_\_\_\_\_

To achieve full marks working and reasoning should be shown.

**This is a *Calculator Assumed Assessment* – 45 minutes / 38 marks**  
**You may have notes on one side of an A4 sheet of paper.**

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1. [ 3 marks]

(I) Suppose a mathematical statement of the form  $P \Rightarrow Q$  is true. Then (Circle your answer)

- (a) Its converse will always be true.
- (b) Its converse will always be false.
- (c) Its negation will be always be true.
- (d) Its contrapositive will always be true. ✓
- (e) Its contrapositive will always be false.

(II) Suppose  $n$  is an integer. Consider the statement: If  $n^2$  is even then  $n$  is even.

The converse of this statement is: (Circle your answer)

- (a) If  $n^2$  is odd, then  $n$  is even.
- (b) If  $n^2$  is even, then  $n$  is odd.
- (c) If  $n^2$  is odd, then  $n$  is odd.
- (d) If  $n$  is odd, then  $n^2$  is odd.
- (e) If  $n$  is even, then  $n^2$  is even. ✓

(III) Consider the statement: The number  $5n^2 - 4n + 1$  is a composite number for every positive integer  $n$ .

The negation of this statement is: (Circle your answer)

- (a) The number  $5n^2 - 4n + 1$  is a prime number for some prime number  $n$ .
- (b) The number  $5n^2 - 4n + 1$  is a prime number for every prime number  $n$ .
- (c) The number  $5n^2 - 4n + 1$  is a prime number for every positive integer  $n$ . ✓
- (d) The number  $5n^2 - 4n + 1$  is a prime number for some positive integer  $n$ .
- (e) The number  $5n^2 - 4n + 1$  is a composite number for some positive integer  $n$ .

2. ~~6~~ [5 marks]

Use mathematical induction to prove that  $7^{2n-1} + 5$  is divisible by 12 for all integers  $n \geq 1$ .

To Prove  $7^{2n-1} + 5$  is divisible by 12 for all integers  $n \geq 1$

Prove for  $n=1$  ie  $7^{2 \times 1 - 1} + 5 = 7 + 5$   
 $= 12$

$\therefore$  true for  $n=1$  ✓

Assume true for  $n=k$

ie  $7^{2k-1} + 5 = 12I$  Where  $I$  is an integer

ie  $7^{2k-1} = 12I - 5$  ✓

Prove true for  $n=k+1$

Now  $7^{2(k+1)-1} + 5 = 7^2 \cdot 7^{2k-1} + 5$  ✓

$= 49(12I - 5) + 5$  (Using Previous)

$= 49 \cdot 12I - 245 + 5$

$= 12(49I) - 240$

$= 12[49I - 20]$  ✓✓

which is divisible by 12

$\therefore$  true for  $n=k+1$

$\therefore$  by mathematical induction, the statement is true for all integers  $n \geq 1$  ✓

3. ~~8~~ [5 marks]

For  $n = 1, 2, 3, \dots$ , let  $S_n = 1^2 + 2^2 + 3^2 \dots + n^2$

Use mathematical induction to prove that, for all integers  $n$  with  $n = 1, 2, 3, \dots$ ,

$$S_n = \frac{1}{6}n(n+1)(2n+1)$$

To Prove  $1^2 + 2^2 + 3^2 \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Proof: For  $n=1$  LHS =  $1^2$  RHS =  $\frac{1}{6}(1)(2)(3)$   
 $= 1$   $= 1$

$\therefore$  true for  $n=1$  ✓

Assume true for  $n=k$

is  $S_k = \frac{1}{6}k(k+1)(2k+1)$  ✓

Need to show true for  $n=k+1$

ie  $S_{k+1} = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$  } either of these ✓

ie  $S_{k+1} = \frac{1}{6}(k+1)(k+2)(2k+3)$

Now  $S_{k+1} = 1^2 + 2^2 + \dots + k^2 + (k+1)^2$  ✓

$= S_k + (k+1)^2$  ✓

$= \frac{1}{6}(k)(k+1)(2k+1) + (k+1)^2$

$= (k+1) \left[ \frac{k(2k+1) + 6(k+1)}{6} \right]$

$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$  ✓

$\therefore$  if true for  $n=k$  then true for  $n=k+1$

$\therefore$  by mathematical induction, true for all natural numbers ✓

4. [6 marks]

Use the fact that if  $n^2$  is divisible by 3, then  $n$  is divisible by 3 to prove that  $\sqrt{3}$  is irrational.

Proof by Contradiction.

Suppose  $\sqrt{3}$  is rational: so  $\sqrt{3} = \frac{p}{q}$  ( $p, q \in \mathbb{Z}$ )

$p$  &  $q$  have no common factors ✓

$$\text{Now } \sqrt{3} = \frac{p}{q}$$

$$\text{ie } \sqrt{3} q = p$$

$$\text{ie } 3q^2 = p^2 \quad \checkmark$$

$\Rightarrow p^2$  is divisible by 3

$\Rightarrow p$  is divisible by 3 ✓

$\Rightarrow p = 3k$  for some  $k \in \mathbb{N}$

$$\text{Hence } (3k)^2 = 3q^2$$

$$\text{ie } q^2 = 3k^2 \quad \checkmark$$

Hence  $q^2$  is divisible by 3

$\Rightarrow q$  is divisible by 3 ✓

◦◦  $p$  and  $q$  have a common factor of 3  
which contradicts that  $p$  &  $q$  have no  
common factors ✓

◦◦  $\sqrt{3}$  is irrational

5. [5 marks]

Suppose that  $a, b \in \mathbb{R}$  and consider the statement: If  $ab$  is irrational then either  $a$  or  $b$  is irrational.

(a) Write down the contrapositive of this statement.

If  $a$  and  $b$  are rational then  $ab$  is rational

(b) Prove the contrapositive of this statement.

Suppose  $a$  and  $b$  are rational

Let  $a = \frac{m}{n}$  and  $b = \frac{p}{q}$   $\left( \begin{array}{l} m, n, p, q \text{ integers} \\ \text{with } n \neq 0 \text{ and } q \neq 0 \end{array} \right)$

Now  $ab = \frac{m}{n} \cdot \frac{p}{q}$

$= \frac{mp}{nq}$  which is rational

6. [5 marks]

From the letters of the word **FACTORISE**, words of 5 letters are arranged without repeating letters.

How many of these arrangements of 5 letters:-

(a) are possible altogether

$${}^9P_5 = 15120 \checkmark$$

(b) begin with the letters **AR** in that particular order



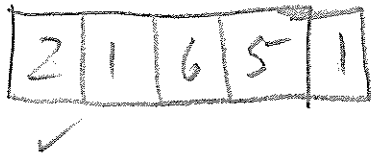
$$\begin{aligned} N^{\circ} \text{ Ways} &= 7 \times 6 \times 5 \\ &= 210 \checkmark \end{aligned}$$

(c) end with the letter **T**



$$\begin{aligned} N^{\circ} \text{ Ways} &= 8 \times 7 \times 6 \times 5 \times 1 \\ &= 1680 \checkmark \end{aligned}$$

(d) start with **AR** in any order (ie **AR** or **RA**) and end with **T**

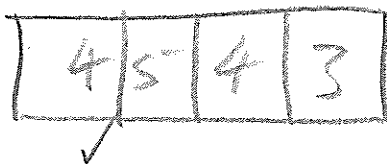


$$\begin{aligned} N^{\circ} \text{ Ways} &= 2 \times 1 \times 6 \times 5 \\ &= 60 \checkmark \end{aligned}$$

7. [5 marks]

Using the digits from this list: 0, 3, 4, 5, 6, 8 determine:

a) How many 4 digit numbers can be made that are greater than 4000?  
(No repetition allowed. You cannot start the number with zero)



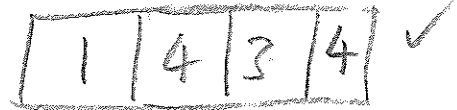
$$\begin{aligned} N^{\circ} \text{ ways} &= 4 \times 5 \times 4 \times 3 \\ &= 240 \checkmark \end{aligned}$$

b) How many 4 digit numbers are even and greater than 4000?  
(No repetition allowed. You cannot start the number with zero)

Starting with 4, 6, 8



Starting with 5



$$\begin{aligned} N^{\circ} \text{ ways} &= 3 \times 4 \times 3 \times 3 & + & & 1 \times 4 \times 3 \times 4 \\ &= 108 & + & & 48 \\ &= 156 \checkmark \end{aligned}$$