PERTH MODERN SCHOOL

YR11 MATHEMATICS SPECIALIST – 2019

TEST 1 – Reasoning & Permutations

NAME: Solutions. DATE:______________

Teacher's Name______________________________

To achieve full marks working and reasoning should be shown.

This is a *Calculator Assumed Assessment – 45 minutes / 38 marks*
You may have notes on one side of an A4 sheet of paper.

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1. **[3 marks]**

(I) Suppose a mathematical statement of the form $P \Rightarrow Q$ is true. Then (Circle your answer)

(a) Its converse will always be true.

(b) Its converse will always be false.

(c) Its negation will be always be true.

(d) Its contrapositive will always be true.

(e) Its contrapositive will always be false.

(II) Suppose n is an integer. Consider the statement: If n^2 is even then n is even.

The converse of this statement is: (Circle your answer)

(a) If n^2 is odd, then n is even.

(b) If n^2 is even, then n is odd.

(c) If n^2 is odd, then n is odd.

(d) If n is odd, then n^2 is odd.

(d) If n is even, then n^2 is even.

(III) Consider the statement: The number $5n^2 - 4n + 1$ is a composite number for every positive integer n.

The negation of this statement is: (Circle your answer)

(a) The number $5n^2 - 4n + 1$ is a prime number for some prime number n.

(b) The number $5n^2 - 4n + 1$ is a prime number for every prime number n.

(c) The number $5n^2 - 4n + 1$ is a prime number for every positive integer n.

(d) The number $5n^2 - 4n + 1$ is a prime number for some positive integer n.

(e) The number $5n^2 - 4n + 1$ is a composite number for some positive integer n.

3

2. ~~[5~~ 6 **marks]**

Use mathematical induction to prove that $7^{2n-1} + 5$ is divisible by 12 for all integers n $\geq$ 1.

To Prove $7^{2n-1} + 5$ is divisible by 12 for all integers $n \geq 1$

Prove for $n=1$ ie $7^{2\times 1-1} + 5 = 7+5$

$= 12$ ✓

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $7^{2k-1} + 5 = 12I$ where I is an integer

ie $7^{2k-1} = 12I - 5$ ✓

Prove true for $n=k+1$

Now $7^{2(k+1)-1} + 5 = 7^2 . 7^{2k-1} + 5$ ✓

$= 49(12I - 5) + 5$ (Using Previous)

$= 49.12I - 245 + 5$

$= 12(49I) - 240$

$= 12[49I - 20]$ ✓✓

which is divisible by 12

$\therefore$ true for $n=k+1$

$\therefore$ by mathematical induction, the statement is true for all integers $n \geq 1$ ✓

3. [~~5~~ 8 marks]

For $n = 1,2,3, \ldots$, let $S_n = 1^2 + 2^2 + 3^2 \ldots + n^2$

Use mathematical induction to prove that, for all integers n with $n = 1,2,3, \ldots$,

$$S_n = \frac{1}{6}n(n+1)(2n+1)$$

To Prove $\quad 1^2 + 2^2 + 3^2 \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1)$

Proof: $\quad$ For $n=1$ $\quad$ LHS $= 1^2 = 1$ $\quad$ RHS $= \frac{1}{6}(1)(2)(3) = 1$

$\therefore$ true for $n=1$ ✓

Assume true for $n=k$

is $\quad S_k = \frac{1}{6} k(k+1)(2k+1)$ ✓

Need to show true for $n=k+1$

ie $\quad S_{k+1} = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

ie $\quad S_{k+1} = \frac{1}{6}(k+1)(k+2)(2k+3)$

✓ either of these

Now $\quad S_{k+1} = 1^2 + 2^2 + \ldots + k^2 + (k+1)^2$ ✓

$$= S_k + (k+1)^2 \quad ✓$$

$$= \frac{1}{6}(k)(k+1)(2k+1) + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right]$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) \quad ✓✓$$

$\therefore$ if true for $n=k$ then true for $n=k+1$

$\therefore$ by mathematical induction, true for all natural numbers ✓

4. **[6 marks]**

Use the fact that if n^2 is divisible by 3, then n is divisible by 3 to prove that $\sqrt{3}$ is irrational.

Proof by Contradiction.

Suppose $\sqrt{3}$ is rational: so $\sqrt{3} = \frac{p}{q}$ $(p, q \in \mathbb{Z})$ ✓

p & q have no common factors

Now $\sqrt{3} = \frac{p}{q}$

ie $\sqrt{3}q = p$

ie $3q^2 = p^2$ ✓

$\Rightarrow p^2$ is divisible by 3

$\Rightarrow p$ is divisible by 3 ✓

$\Rightarrow p = 3k$ for some $k \in \mathbb{N}$

Hence $(3k)^2 = 3q^2$

ie $q^2 = 3k^2$ ✓

Hence q^2 is divisible by 3

$\Rightarrow q$ is divisible by 3 ✓

$\therefore$ p and q have a common factor of 3

which contradicts that p & q have no common factors ✓

$\therefore \sqrt{3}$ is irrational

5. **[5 marks]**

Suppose that a, $b \in \mathbb{R}$ and consider the statement: If ab is irrational then either a or b is irrational.

(a) Write down the contrapositive of this statement.

If a and b are rational then ab is rational

(b) Prove the contrapositive of this statement.

Suppose a and b are rational

Let $a = \frac{m}{n}$ and $b = \frac{p}{q}$ (m, n, p, q integers with $n \neq 0$ and $q \neq 0$)

Now $ab = \frac{m}{n} \cdot \frac{p}{q}$

$= \frac{mp}{nq}$ which is rational

6. **[5 marks]**

From the letters of the word **FACTORISE**, words of 5 letters are arranged without repeating letters.

How many of these arrangements of 5 letters:-

(a) are possible altogether $^9P_5 = 15120$

(b) begin with the letters **AR** in that particular order

1	1	7	6	5

$N^{\underline{o}}$ Ways $= 7 \times 6 \times 5$
$= 210$

(c) end with the letter **T**

8	7	6	5	1

$N^{\underline{o}}$ Ways $= 8 \times 7 \times 6 \times 5 \times 1$
$= 1680$

(d) start with **AR** in any order (ie **AR** or **RA**) and end with **T**

2	1	6	5	1

$N^{\underline{o}}$ Ways $= 2 \times 1 \times 6 \times 5$
$= 60$

7. **[5 marks]**

Using the digits from this list: 0, 3, 4, 5, 6, 8 determine:

a) How many 4 digit numbers can be made that are greater than 4000?
(No repetition allowed. You cannot start the number with zero)

4	5	4	3

$N^{\underline{o}}$ ways $= 4 \times 5 \times 4 \times 3$
$= 240$

b) How many 4 digit numbers are even and greater than 4000?
(No repetition allowed. You cannot start the number with zero)

Starting with 4, 6, 8

3	4	3	3

Starting with 5

1	4	3	4

$N^{\underline{o}}$ ways $= 3 \times 4 \times 3 \times 3 + 1 \times 4 \times 3 \times 4$
$= 108 + 48$
$= 156$